第 4 章：過渡現象

4.4　過渡現象の解法

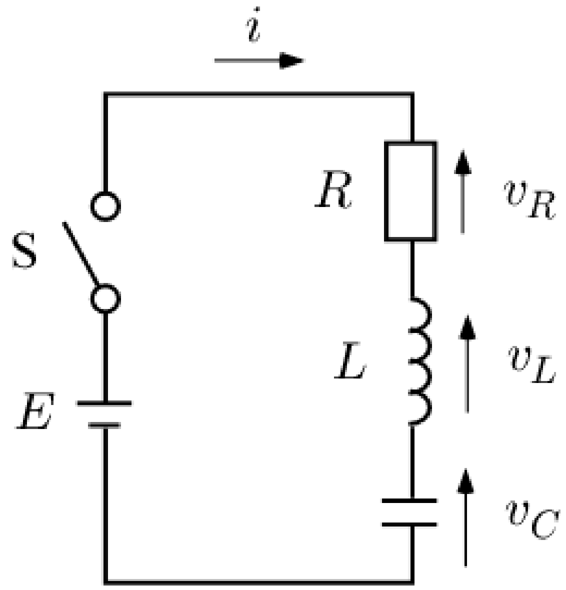
キーワード：RLC直列回路

学習目標：RLC直列回路の過渡現象を解くことができるようになる。

4 過渡現象

4.4.5 *RLC*直列回路

［問題46］



$$L\frac{di}{dt} + Ri + \frac{1}{C}\int i\ dt = E$$

$i = \dfrac{dq}{dt}$ の関係を用いると

$$L\frac{d^2q}{dt} + R\frac{dq}{dt} + \frac{1}{C}q = E$$

定常解

$$L\underbrace{\frac{d^2q_s}{dt}}_{=0} + R\underbrace{\frac{dq_s}{dt}}_{=0} + \frac{1}{C}q_s = E$$

$$\frac{1}{C}q_s = E$$

$$q_s = CE$$

過渡解

$$L\frac{d^2q_t}{dt}+R\frac{dq_t}{dt}+\frac{1}{C}q_t=0$$

$$\frac{dq_t}{dt}=Ape^{pt}$$

$$\frac{d^2q_t}{dt^2}=Ap^2e^{pt}$$

$q_t=Ae^{pt}$ とおく

$$LAp^2e^{pt}+RApe^{pt}+\frac{1}{C}Ae^{pt}=0$$

$$\left(Lp^2+Rp+\frac{1}{C}\right)Ae^{pt}=0$$

よって

$$Lp^2+Rp+\frac{1}{C}=0$$

が成り立つ

$$p_1,p_2=\frac{-R\pm\sqrt{R^2-4\frac{L}{C}}}{2L}$$

$R^2-4\frac{L}{C}$

>0 (1) 異なる2実数根と持つ場合

$=0$ (2) 重根を持つ場合

<0 (3) 虚数根を持つ場合

$$p_1, p_2 = \frac{-R \pm \sqrt{R^2 - 4\frac{L}{C}}}{2L} = -\frac{R}{2L} \pm \sqrt{\frac{R^2}{(2L)^2} - 4\frac{L}{C(2L)^2}}$$

$$= -\frac{R}{2L} \pm \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}$$

$\frac{R}{2L} = \alpha$, $\left(\frac{R}{2L}\right)^2 = \alpha^2$, $\frac{1}{LC} = \omega_0^2$

$$\alpha = \frac{R}{2L}, \ \omega_0 = \frac{1}{\sqrt{LC}}$$

(1) 異なる2実数根と持つ場合 $\left(R^2 > \frac{4L}{C}\right)$

$$p_1, p_2 = -\frac{R}{2L} \pm \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2}$$

$$q_t = A_1 e^{p_1 t} + A_2 e^{p_2 t}$$

$$= A_1 e^{\left(-\alpha + \sqrt{\alpha^2 - \omega_0^2}\right)t} + A_2 e^{\left(-\alpha - \sqrt{\alpha^2 - \omega_0^2}\right)t}$$

$$= e^{-\alpha t}\left(A_1 e^{\left(\sqrt{\alpha^2 - \omega_0^2}\right)t} + A_2 e^{\left(-\sqrt{\alpha^2 - \omega_0^2}\right)t}\right)$$

一般解

$$q = q_s + q_t \qquad \text{(定常解+過渡解)}$$

$$= CE + e^{-\alpha t}\left(A_1 e^{\left(\sqrt{\alpha^2-\omega_0^2}\right)t} + A_2 e^{\left(-\sqrt{\alpha^2-\omega_0^2}\right)t}\right) \qquad \text{(1-1)}$$

$$i = \frac{dq}{dt} = -\alpha e^{-\alpha t}\left(A_1 e^{\left(\sqrt{\alpha^2-\omega_0^2}\right)t} + A_2 e^{\left(-\sqrt{\alpha^2-\omega_0^2}\right)t}\right)$$

$$+e^{-\alpha t}\left(A_1\sqrt{\alpha^2-\omega_0^2}e^{\left(\sqrt{\alpha^2-\omega_0^2}\right)t} - A_2\sqrt{\alpha^2-\omega_0^2}e^{\left(-\sqrt{\alpha^2-\omega_0^2}\right)t}\right)$$

$$= e^{-\alpha t}\left[A_1\left(-\alpha+\sqrt{\alpha^2-\omega_0^2}\right)e^{\left(\sqrt{\alpha^2-\omega_0^2}\right)t} + A_2\left(-\alpha-\sqrt{\alpha^2-\omega_0^2}\right)e^{\left(-\sqrt{\alpha^2-\omega_0^2}\right)t}\right]$$

(1-2)

初期条件　$t=0,\ q=0,\ i=0$

(1-1)式に $t=0,\ q=0$ を代入

$$q = CE + e^{-\alpha t}\left(A_1 e^{\left(\sqrt{\alpha^2-\omega_0^2}\right)t} + A_2 e^{\left(-\sqrt{\alpha^2-\omega_0^2}\right)t}\right) \quad \text{(1-1)}$$

$$0 = CE + e^0\left(A_1 e^0 + A_2 e^0\right)$$

$$A_1 + A_2 = -CE$$

$$A_2 = -A_1 - CE \qquad \text{(1-3)}$$

(1-2)式に $t=0,\ i=0$ を代入

$$i = e^{-\alpha t}\left[A_1\left(-\alpha+\sqrt{\alpha^2-\omega_0^2}\right)e^{\left(\sqrt{\alpha^2-\omega_0^2}\right)t} + A_2\left(-\alpha-\sqrt{\alpha^2-\omega_0^2}\right)e^{\left(-\sqrt{\alpha^2-\omega_0^2}\right)t}\right] \quad \text{(1-2)}$$

$$0 = e^0\left[A_1\left(-\alpha+\sqrt{\alpha^2-\omega_0^2}\right)e^0 + A_2\left(-\alpha-\sqrt{\alpha^2-\omega_0^2}\right)e^0\right]$$

$$0 = A_1\left(-\alpha+\sqrt{\alpha^2-\omega_0^2}\right) + A_2\left(-\alpha-\sqrt{\alpha^2-\omega_0^2}\right)$$

$$0=-\alpha\left(A_1+A_2\right)+\left(A_1\sqrt{\alpha^2-\omega_0^2}-A_2\sqrt{\alpha^2-\omega_0^2}\right)$$

$$0=CE\alpha+\left(A_1\sqrt{\alpha^2-\omega_0^2}-(-A_1-CE)\sqrt{\alpha^2-\omega_0^2}\right)$$

$$0=CE\alpha+\left(A_1\sqrt{\alpha^2-\omega_0^2}-(-A_1-CE)\sqrt{\alpha^2-\omega_0^2}\right)$$

$$0=\alpha CE+2A_1\sqrt{\alpha^2-\omega_0^2}+CE\sqrt{\alpha^2-\omega_0^2}$$

$$2A_1\sqrt{\alpha^2-\omega_0^2}=-CE\left(\alpha+\sqrt{\alpha^2-\omega_0^2}\right)$$

$$A_1=-CE\left(\frac{\alpha}{2\sqrt{\alpha^2-\omega_0^2}}+\frac{1}{2}\right) \quad \text{(1-4)}$$

(1-3)式に(1-4)式を代入

$$A_2=-A_1-CE \quad \text{(1-3)}$$

$$A_2=CE\left(\frac{\alpha}{2\sqrt{\alpha^2-\omega_0^2}}+\frac{1}{2}\right)-CE=CE\left(\frac{\alpha}{2\sqrt{\alpha^2-\omega_0^2}}-\frac{1}{2}\right) \quad \text{(1-5)}$$

$$A_1 = -CE\left(\frac{\alpha}{2\sqrt{\alpha^2-\omega_0^2}} + \frac{1}{2}\right) \quad \text{(1-4)}$$

$$A_2 = CE\left(\frac{\alpha}{2\sqrt{\alpha^2-\omega_0^2}} - \frac{1}{2}\right) \quad \text{(1-5)}$$

$$q = CE + e^{-\alpha t}\left(A_1 e^{\left(\sqrt{\alpha^2-\omega_0^2}\right)t} + A_2 e^{\left(-\sqrt{\alpha^2-\omega_0^2}\right)t}\right) \quad \text{(1-1)}$$

(1-4), (1-5)式を(1-1)式に代入

$$q = CE + e^{-\alpha t}\left(-CE\left(\frac{\alpha}{2\sqrt{\alpha^2-\omega_0^2}} + \frac{1}{2}\right)e^{\left(\sqrt{\alpha^2-\omega_0^2}\right)t}\right.$$
$$\left. + CE\left(\frac{\alpha}{2\sqrt{\alpha^2-\omega_0^2}} - \frac{1}{2}\right)e^{\left(-\sqrt{\alpha^2-\omega_0^2}\right)t}\right)$$

$$= CE\left[1 + e^{-\alpha t}\left\{-\left(\frac{\alpha}{2\sqrt{\alpha^2-\omega_0^2}} + \frac{1}{2}\right)e^{\left(\sqrt{\alpha^2-\omega_0^2}\right)t} + \left(\frac{\alpha}{2\sqrt{\alpha^2-\omega_0^2}} - \frac{1}{2}\right)e^{\left(-\sqrt{\alpha^2-\omega_0^2}\right)t}\right\}\right]$$

$$q = CE\left[1 + e^{-\alpha t}\left\{-\left(\frac{\alpha}{2\sqrt{\alpha^2-\omega_0^2}} + \frac{1}{2}\right)e^{\left(\sqrt{\alpha^2-\omega_0^2}\right)t} + \left(\frac{\alpha}{2\sqrt{\alpha^2-\omega_0^2}} - \frac{1}{2}\right)e^{\left(-\sqrt{\alpha^2-\omega_0^2}\right)t}\right\}\right]$$

$$= CE\left[1 + \frac{e^{-\alpha t}}{2}\left\{-\left(\frac{\alpha}{\sqrt{\alpha^2-\omega_0^2}} + 1\right)e^{\left(\sqrt{\alpha^2-\omega_0^2}\right)t} + \left(\frac{\alpha}{\sqrt{\alpha^2-\omega_0^2}} - 1\right)e^{\left(-\sqrt{\alpha^2-\omega_0^2}\right)t}\right\}\right]$$

電流

$$i = \frac{dq}{dt}$$

$$i = \frac{CE}{2}\left[-\alpha e^{-\alpha t}\left\{-\left(\frac{\alpha}{\sqrt{\alpha^2-\omega_0^2}} + 1\right)e^{\left(\sqrt{\alpha^2-\omega_0^2}\right)t} + \left(\frac{\alpha}{\sqrt{\alpha^2-\omega_0^2}} - 1\right)e^{\left(-\sqrt{\alpha^2-\omega_0^2}\right)t}\right\}\right.$$

$$\left. e^{-\alpha t}\left\{-\left(\alpha + \sqrt{\alpha^2-\omega_0^2}\right)e^{\left(\sqrt{\alpha^2-\omega_0^2}\right)t} + \left(-\alpha + \sqrt{\alpha^2-\omega_0^2}\right)e^{\left(-\sqrt{\alpha^2-\omega_0^2}\right)t}\right\}\right]$$

$$= \frac{CE}{2}\left[e^{-\alpha t}\left\{\left(\frac{\alpha^2}{\sqrt{\alpha^2-\omega_0^2}} - \sqrt{\alpha^2-\omega_0^2}\right)e^{\left(\sqrt{\alpha^2-\omega_0^2}\right)t} + \left(\frac{-\alpha^2}{\sqrt{\alpha^2-\omega_0^2}} + \sqrt{\alpha^2-\omega_0^2}\right)e^{\left(-\sqrt{\alpha^2-\omega_0^2}\right)t}\right\}\right.$$

$$= \frac{\omega_0^2}{\sqrt{\alpha^2-\omega_0^2}} \qquad = -\frac{\omega_0^2}{\sqrt{\alpha^2-\omega_0^2}}$$

$$i = \frac{CE}{2}\left[e^{-\alpha t}\left\{\frac{\omega_0^2}{\sqrt{\alpha^2-\omega_0^2}}e^{\left(\sqrt{\alpha^2-\omega_0^2}\right)t} - \frac{\omega_0^2}{\sqrt{\alpha^2-\omega_0^2}}e^{\left(-\sqrt{\alpha^2-\omega_0^2}\right)t}\right\}\right.$$

$$= \frac{CE}{2}\frac{\omega_0^2}{\sqrt{\alpha^2-\omega_0^2}}\left[e^{-\alpha t}\left\{e^{\left(\sqrt{\alpha^2-\omega_0^2}\right)t} - e^{\left(-\sqrt{\alpha^2-\omega_0^2}\right)t}\right\}\right.$$

$$= \frac{\frac{1}{LC}}{\sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}} = \frac{\frac{1}{C}}{\sqrt{\left(\frac{R}{2}\right)^2 - \frac{L}{C}}}$$

$$= \frac{CE}{2}\frac{\omega_0^2}{\sqrt{\alpha^2-\omega_0^2}}e^{-\alpha t}\left\{e^{\left(\sqrt{\alpha^2-\omega_0^2}\right)t} - e^{\left(-\sqrt{\alpha^2-\omega_0^2}\right)t}\right\}$$

$$= \frac{CE}{2}\frac{\frac{1}{C}}{\sqrt{\left(\frac{R}{2}\right)^2 - \frac{L}{C}}}e^{-\alpha t}\left\{e^{\left(\sqrt{\alpha^2-\omega_0^2}\right)t} - e^{\left(-\sqrt{\alpha^2-\omega_0^2}\right)t}\right\}$$

$$= \frac{E}{\sqrt{\left(\frac{R}{2}\right)^2 - \frac{L}{C}}}e^{-\alpha t}\frac{e^{\left(\sqrt{\alpha^2-\omega_0^2}\right)t} - e^{\left(-\sqrt{\alpha^2-\omega_0^2}\right)t}}{2}$$

$$= \frac{E}{\sqrt{\left(\frac{R}{2}\right)^2 - \frac{L}{C}}}e^{-\alpha t}\sinh\left(\sqrt{\alpha^2-\omega_0^2}\right)t$$

【問題41.4】

(3) 虚数根を持つ場合 $\left(R^2 < \frac{4L}{C}\right)$

$$p_1, p_2 = \frac{-R \pm \sqrt{R^2 - 4\frac{L}{C}}}{2L} = \frac{-R \pm j\sqrt{4\frac{L}{C} - R^2}}{2L}$$

$$= -\frac{R}{2L} \pm j\sqrt{\frac{1}{LC} - \left(\frac{R}{2L}\right)^2} = -\alpha \pm j\omega$$

$$q_t = A_1 e^{p_1 t} + A_2 e^{p_2 t}$$

$$= A_1 e^{(-\alpha + j\omega)t} + A_2 e^{(-\alpha - j\omega)t}$$

$$= e^{-\alpha t}(A_1 e^{j\omega t} + A_2 e^{-j\omega t})$$

一般解

$$q = q_s + q_t$$

$$= CE + e^{-\alpha t}(A_1 e^{j\omega t} + A_2 e^{-j\omega t})$$

$$= CE + e^{-\alpha t}\left[A_1(\cos\omega t + j\sin\omega t) + A_2(\cos\omega t - j\sin\omega t)\right]$$

$$= CE + e^{-\alpha t}\left[(A_1 + A_2)\cos\omega t + j(A_1 - A_2)\sin\omega t)\right] \quad \text{(2-1)}$$

電流

$$i = \frac{dq}{dt} = -\alpha e^{-\alpha t}\left((A_1 + A_2)\cos\omega t + j(A_1 - A_2)\sin\omega t\right)$$
$$+ e^{-\alpha t}\left(-(A_1 + A_2)\omega\sin\omega t + j\omega(A_1 - A_2)\cos\omega t\right)$$

$$= e^{-\alpha t}\left[\{-\alpha(A_1 + A_2) + j\omega(A_1 - A_2)\}\cos\omega t\right.$$
$$\left.+ \{-(A_1 + A_2)\omega - j\alpha(A_1 - A_2)\}\sin\omega t\right] \quad \text{(2-2)}$$

初期条件　$t=0,\ q=0,\ i=0$

(2-1)式に $t=0,\ q=0$ を代入

$$q = CE + e^{-\alpha t}\left[(A_1+A_2)\cos\omega t + j(A_1-A_2)\sin\omega t)\right] \qquad \text{(2-1)}$$

$$0 = CE + e^{0}\left[(A_1+A_2)\cos 0 + j(A_1-A_2)\underbrace{\sin 0}_{=0})\right]$$

$$A_1 + A_2 = -CE \qquad \text{(2-3)}$$

(2-2)式に $t=0,\ i=0$ を代入

$$\begin{aligned} i = e^{-\alpha t}\,[&\{-\alpha(A_1+A_2) + j\omega(A_1-A_2)\}\cos\omega t \\ &+\{-(A_1+A_2)\omega - j\alpha(A_1-A_2)\}\sin\omega t] \qquad \text{(2-2)}\end{aligned}$$

$$\begin{aligned} 0 = e^{0}\,[&\{-\alpha(A_1+A_2) + j\omega(A_1-A_2)\}\cos 0 \\ &+\{-(A_1+A_2)\omega - j\alpha(A_1-A_2)\}\underbrace{\sin 0}_{=0}]\end{aligned}$$

$$0 = -\alpha(A_1+A_2) + j\omega(A_1-A_2) \qquad \text{(2-4)}$$

$$0 = -\alpha(A_1 + A_2) + j\omega(A_1 - A_2) \qquad \text{(2-4)}$$

$$j\omega(A_1 - A_2) = \alpha(A_1 + A_2)$$

$$j(A_1 - A_2) = \frac{\alpha(A_1 + A_2)}{\omega} = \frac{\alpha(-CE)}{\omega} \qquad \text{(2-5)}$$

(2-1)式に(2-4)式，(2-5)式を代入

$$\begin{aligned} q &= CE + e^{-\alpha t}\left[(A_1 + A_2)\cos\omega t + j(A_1 - A_2)\sin\omega t)\right] \\ &= CE + e^{-\alpha t}\left[-CE\cos\omega t - \frac{\alpha CE}{\omega}\sin\omega t)\right] \\ &= CE\left[1 - e^{-\alpha t}\left(\cos\omega t + \frac{\alpha}{\omega}\sin\omega t)\right)\right] \\ &= CE\left[1 - e^{-\alpha t}\left(\frac{\alpha}{\omega}\sin\omega t + \cos\omega t\right)\right] \qquad \text{(2-6)} \end{aligned}$$

$$q = CE\left[1 - e^{-\alpha t}\left(\frac{\alpha}{\omega}\sin\omega t + \cos\omega t\right)\right] \quad \text{(2-6)}$$

$$\frac{\alpha}{\omega}\sin\omega t + \cos\omega t = \sqrt{\left(\frac{\alpha}{\omega}\right)^2 + 1^2}\cdot\sin(\omega t + \theta_1)$$

$$\tan\theta_1 = \frac{1}{\frac{\alpha}{\omega}} = \frac{\omega}{\alpha}$$

【公式】$a\sin\omega t + b\cos\omega t = \sqrt{a^2 + b^2}\sin(\omega t + \theta)$

$$\tan\omega t = \frac{b}{a}$$

ここで

$$\left(\frac{\alpha}{\omega}\right)^2 = \frac{\left(\frac{R}{2L}\right)^2}{\frac{1}{LC} - \left(\frac{R}{2L}\right)^2} = \frac{1}{\frac{1}{LC}\frac{4L^2}{R^2} - 1} = \frac{1}{\frac{4L}{CR^2} - 1}$$

より

$$\sqrt{\left(\frac{\alpha}{\omega}\right)^2 + 1^2} = \sqrt{\frac{1}{\frac{4L}{CR^2} - 1} + 1} = \sqrt{\frac{\frac{4L}{CR^2}}{\frac{4L}{CR^2} - 1}} = \sqrt{\frac{1}{\frac{CR^2}{4L} - 1}}$$

$$q = CE\left[1 - e^{-\alpha t}\frac{1}{\sqrt{\frac{CR^2}{4L} - 1}}\sin(\omega t + \theta_1)\right]$$

$$q = CE\left[1 - e^{-\alpha t}\frac{1}{\sqrt{\frac{CR^2}{4L} - 1}}\sin(\omega t + \theta_1)\right]$$

$$i = \frac{dq}{dt}$$

$$= CE\left[\alpha e^{-\alpha t}\frac{1}{\sqrt{\frac{CR^2}{4L} - 1}}\sin(\omega t + \theta_1) - e^{-\alpha t}\frac{1}{\sqrt{\frac{CR^2}{4L} - 1}}\omega\cos(\omega t + \theta_1)\right]$$

$$= CE\frac{1}{\sqrt{\frac{CR^2}{4L} - 1}}e^{-\alpha t}\left[\alpha\sin(\omega t + \theta_1) - \omega\cos(\omega t + \theta_1)\right]$$

$$= CE\frac{1}{\sqrt{\frac{CR^2}{4L} - 1}}e^{-\alpha t}\left[\sqrt{\alpha^2 + \omega^2}\sin(\omega t + \theta_1 + \theta_2)\right]$$

$$\tan\theta_1 = \frac{\omega}{\alpha}$$

$$\tan\theta_2 = -\frac{\omega}{\alpha}$$

ここで

$$\tan(\theta_1+\theta_2) = \frac{\tan\theta_1+\tan\theta_2}{1-\tan\theta_1\tan\theta_2} = 0$$

$$\tan\theta_1 = \frac{\omega}{\alpha}$$

$$\tan\theta_2 = -\frac{\omega}{\alpha}$$

より

$$\theta_1+\theta_2 = 0$$

よって

$$i = CE\frac{1}{\sqrt{\frac{CR^2}{4L}-1}}e^{-\alpha t}\left[\sqrt{\alpha^2+\omega^2}\sin(\omega t)\right]$$

【問題41.5】

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