

第 24 章 : 非正弦波交流

- 24.1 非正弦波交流
- 24.2 正弦波の組み合わせと波形
- 24.3 フーリエ級数による非正弦波の展開

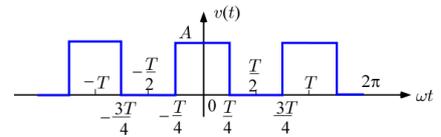
キーワード : 対称波, 奇関数波, 偶関数波

学習目標 : 対称波, 奇関数波, 偶関数波のフーリエ級数を計算できる。

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[問題30.1](1)

$$\omega = \frac{2\pi}{T}$$



$$f(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\omega t + \sum_{n=1}^{\infty} b_n \sin n\omega t$$

偶関数より $b_n = 0$

$$b_n = \frac{2}{T} \int_0^T v(t) \sin n\omega t dt$$

a_0 を求める

(解法1)

$$a_0 = \frac{1}{T} \int_0^T v(t) dt = \frac{1}{T} \left(\int_0^{T/4} A dt + \int_{3T/4}^T A dt \right)$$

(解法2) 半周期を2倍

$$a_0 = \frac{2}{T} \int_0^{T/2} v(t) dt = \frac{2}{T} \int_0^{T/4} A dt$$

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(解法3) 周期を $-\frac{T}{4} \sim \frac{3T}{4}$ と考える

$$a_0 = \frac{1}{T} \int_{-\frac{T}{4}}^{\frac{3T}{4}} v(t) dt = \frac{1}{T} \int_{-\frac{T}{4}}^{\frac{T}{4}} A dt$$

(解法1) で解く

$$a_0 = \frac{A}{T} \left([t]_0^{T/4} + [t]_{3T/4}^T \right) = \frac{A}{T} \left(\frac{T}{4} + \frac{T}{4} \right) = \frac{A}{2}$$

(解法2) で解く

$$a_0 = \frac{2A}{T} [t]_0^{T/4} = \frac{2AT}{T \cdot 4} = \frac{A}{2}$$

(解法3) で解く

$$a_0 = \frac{A}{T} [t]_{-\frac{T}{4}}^{\frac{T}{4}} = \frac{A}{T} \left(\frac{T}{4} + \frac{T}{4} \right) = \frac{A}{2}$$

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a_n を求める

(解法1) で解く

$$\begin{aligned} a_n &= \frac{2}{T} \int_0^T v(t) \cos n\omega t dt \\ &= \frac{2}{T} \left(\int_0^{T/4} A \cos n\omega t dt + \int_{3T/4}^T A \cos n\omega t dt \right) \\ &= \frac{2A}{T} \left(\left[\frac{1}{n\omega} \sin n\omega t \right]_0^{T/4} + \left[\frac{1}{n\omega} \sin n\omega t \right]_{3T/4}^T \right) \quad \omega T = 2\pi \text{ より} \\ &= \frac{2A}{n\omega T} \left(\sin n\omega \frac{T}{4} - \sin 0 + \sin n\omega T - \sin n\omega \frac{3T}{4} \right) \end{aligned}$$

$\omega T = 2\pi$ より

$$\begin{aligned} a_n &= \frac{2A}{2\pi n} \left(\sin \frac{2\pi n}{4} + \sin \frac{2\pi n}{2} - \sin \frac{6\pi n}{4} \right) \\ &= \frac{A}{\pi n} \left(\sin \frac{\pi n}{2} - \sin \frac{3\pi n}{2} \right) = \frac{2A}{\pi n} \sin \frac{\pi n}{2} \\ &= \sin \frac{\pi n}{2} \end{aligned}$$

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$$a_1 = \frac{2A}{\pi} \sin \frac{\pi}{2} = \frac{2A}{\pi}$$

$$a_2 = \frac{2A}{2\pi} \sin \frac{2\pi}{2} = 0$$

$$a_3 = \frac{2A}{3\pi} \sin \frac{3\pi}{2} = -\frac{2A}{3\pi}$$

よって

$$v(t) = \frac{A}{2} + \frac{2A}{\pi} \left(1 \cdot \cos \omega t - \frac{1}{3} \cos 3\omega t + \frac{1}{5} \cos 5\omega t + \dots \right)$$

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$$b_n = \frac{2}{T} \int_0^T v(t) \sin n\omega t dt$$

$$= \frac{2}{T} \left(\int_0^{T/4} A \sin n\omega t dt + \int_{3T/4}^T A \sin n\omega t dt \right)$$

$$= \frac{2A}{T} \left(\left[-\frac{1}{n\omega} \cos n\omega t \right]_0^{T/4} + \left[-\frac{1}{n\omega} \cos n\omega t \right]_{3T/4}^T \right)$$

$$= -\frac{2A}{n\omega T} \left(\cos n\omega \frac{T}{4} - \cos 0 + \cos n\omega T - \cos n\omega \frac{3T}{4} \right)$$

$$= -\frac{2A}{2n\pi} \left(\cos n\frac{\pi}{2} - \cos 0 + \cos 2n\pi - \cos n\frac{3\pi}{2} \right)$$

$$\cos n\frac{\pi}{2} = 0, \quad (n = 1, 2, \dots) \quad \cos n\frac{3\pi}{2} = 0, \quad (n = 1, 2, \dots)$$

$$\cos 2n\pi = 1, \quad (n = 1, 2, \dots)$$

$$= 0$$

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(解法2)で解く * (解法1)との違いを示す

$$a_n = \frac{2}{T} \int_0^T v(t) \cos n\omega t dt$$

$$= \frac{4}{T} \left(\int_0^{\frac{T}{4}} A \cos n\omega t dt + \int_{\frac{3T}{4}}^T A \cos n\omega t dt \right)$$

$$= \frac{4}{T} \left(\left[\frac{1}{n\omega} \sin n\omega t \right]_0^{\frac{T}{4}} + \left[\frac{1}{n\omega} \sin n\omega t \right]_{\frac{3T}{4}}^T \right)$$

$$= \frac{4}{n\omega T} \left(\sin n\omega \frac{T}{4} - \sin 0 + \sin n\omega T - \sin n\omega \frac{3T}{4} \right)$$

$\omega T = 2\pi$ より

$$a_n = \frac{4}{2\pi n} \left(\sin \frac{2\pi n}{4} + \sin 2\pi n - \sin \frac{6\pi n}{4} \right)$$

$$= \frac{2}{\pi n} \left(\sin \frac{\pi n}{2} + \sin \frac{3\pi n}{2} \right) = \frac{2A}{\pi n} \sin \frac{\pi n}{2}$$

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(解法3)で解く * (解法1)との違いを示す

$$a_n = \frac{2}{T} \int_0^T v(t) \cos n\omega t dt$$

$$= \frac{2}{T} \left(\int_{-\frac{T}{4}}^{\frac{T}{4}} A \cos n\omega t dt + \int_{\frac{3T}{4}}^T A \cos n\omega t dt \right)$$

$$= \frac{2A}{T} \left(\left[\frac{1}{n\omega} \sin n\omega t \right]_{-\frac{T}{4}}^{\frac{T}{4}} + \left[\frac{1}{n\omega} \sin n\omega t \right]_{\frac{3T}{4}}^T \right)$$

$$= \frac{2A}{n\omega T} \left(\sin n\omega \frac{T}{4} - \sin \left(-n\omega \frac{T}{4} \right) + \sin n\omega T - \sin n\omega \frac{3T}{4} \right)$$

$\omega T = 2\pi$ より

$$a_n = \frac{4}{2\pi n} \left(\sin \frac{2\pi n}{4} + \sin 2\pi n - \sin \frac{6\pi n}{4} \right)$$

$$= \frac{2}{\pi n} \left(\sin \frac{\pi n}{2} + \sin \frac{3\pi n}{2} \right) = \frac{2A}{\pi n} \sin \frac{\pi n}{2}$$

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[問題30.1] (2)

$\omega = \frac{2\pi}{T}$

$f(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\omega t + \sum_{n=1}^{\infty} b_n \sin n\omega t$

奇関数より $a_0 = 0, a_n = 0$

b_n を求める

(解法1)

$$b_n = \frac{2}{T} \int_0^T v(t) \sin n\omega t dt$$

$$= \frac{2}{T} \left(\int_0^{\frac{T}{2}} \frac{2A}{T} t \sin n\omega t dt + \int_{\frac{T}{2}}^T \left(\frac{2A}{T} t - 2A \right) \sin n\omega t dt \right)$$

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(解法2)半周期を2倍

$$b_n = \frac{4}{T} \int_0^{\frac{T}{2}} v(t) \sin n\omega t dt = \frac{4}{T} \int_0^{\frac{T}{2}} \frac{2A}{T} t \sin n\omega t dt$$

(解法3)周期を $-\frac{T}{2} \sim \frac{T}{2}$ と考える

$$b_n = \frac{2}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} v(t) \sin n\omega t dt = \frac{2}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} \frac{2A}{T} t \sin n\omega t dt$$

b_n を求める

(解法2)で解く

$$b_n = \frac{4}{T} \int_0^{\frac{T}{2}} \frac{2A}{T} t \sin n\omega t dt$$

$$= \frac{8A}{T^2} \int_0^{\frac{T}{2}} t \left(-\frac{1}{n\omega} \cos n\omega t \right)' dt$$

$$= \frac{8A}{T^2} \int_0^{\frac{T}{2}} t \sin n\omega t dt$$

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部分積分を用いる

$$b_n = \frac{8A}{T^2} \left\{ \left[t \left(-\frac{1}{n\omega} \cos n\omega t \right) \right]_0^{\frac{T}{2}} - \int_0^{\frac{T}{2}} -\frac{1}{n\omega} \cos n\omega t dt \right\}$$

$$= -\frac{8A}{n\omega T^2} \left\{ \left[t \cos n\omega t \right]_0^{\frac{T}{2}} - \int_0^{\frac{T}{2}} \cos n\omega t dt \right\}$$

$$= -\frac{8A}{n\omega T^2} \left\{ \frac{T}{2} \cos n\omega \frac{T}{2} - 0 - \left[\frac{1}{n\omega} \sin n\omega t \right]_0^{\frac{T}{2}} \right\}$$

$\omega T = 2\pi$ より

$$b_n = -\frac{8A}{n2\pi T} \left\{ \frac{T}{2} \cos n\pi - \frac{1}{n\omega} (\sin n\pi) \right\}$$

$$= -\frac{2A}{n\pi} \cos n\pi$$

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$$b_1 = -\frac{2A}{\pi} \cos \pi = \frac{2A}{\pi}$$

$$b_2 = -\frac{2A}{2\pi} \cos 2\pi = -\frac{2A}{2\pi}$$

$$b_3 = -\frac{2A}{3\pi} \cos 3\pi = \frac{2A}{3\pi}$$

よって

$$v(t) = \frac{2A}{\pi} \left(\sin \omega t - \frac{1}{2} \sin 2\omega t + \frac{1}{3} \sin 3\omega t + \dots \right)$$

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[問題30.1](3)

$$\omega = \frac{2\pi}{T}$$

$$f(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\omega t + \sum_{n=1}^{\infty} b_n \sin n\omega t$$

偶関数より $b_n = 0$

a_0 を求める

(解法1)

$$a_0 = \frac{1}{T} \int_0^T v(t) dt = \frac{1}{T} \left(\int_0^{T/2} A \sin \omega t dt + \int_{T/2}^T (-A \sin \omega t) dt \right)$$

(解法2) 半周期を2倍

$$a_0 = \frac{2}{T} \int_0^{T/2} v(t) dt = \frac{2}{T} \int_0^{T/2} A \sin \omega t dt$$

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(解法1) で解く

$$a_0 = \frac{1}{T} \left(\int_0^{T/2} A \sin \omega t dt + \int_{T/2}^T (-A \sin \omega t) dt \right)$$

$$= \frac{A}{T} \left(\left[-\frac{1}{\omega} \cos \omega t \right]_0^{T/2} + \left[\frac{1}{\omega} \cos \omega t \right]_{T/2}^T \right)$$

$$= \frac{A}{\omega T} \left([-\cos \omega t]_0^{T/2} + [\cos \omega t]_{T/2}^T \right)$$

$$= \frac{A}{\omega T} \left(-\cos \frac{\omega T}{2} + \cos 0 + \cos \omega T - \cos \frac{\omega T}{2} \right)$$

$$= \frac{A}{2\pi} (-\cos \pi + 1 + \cos 2\pi - \cos \pi)$$

$$= \frac{A}{2\pi} (1 + 1 + 1 + 1)$$

$$= \frac{2A}{\pi}$$

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(解法2) で解く

$$a_0 = \frac{2}{T} \int_0^{T/2} A \sin \omega t dt$$

$$= \frac{2A}{T} \left(\left[-\frac{1}{\omega} \cos \omega t \right]_0^{T/2} \right)$$

$$= \frac{2A}{\omega T} \left([-\cos \omega t]_0^{T/2} \right)$$

$$= \frac{2A}{\omega T} \left(-\cos \frac{\omega T}{2} + \cos 0 \right)$$

$$= \frac{2A}{2\pi} (-\cos \pi + 1)$$

$$= \frac{2A}{2\pi} (1 + 1)$$

$$= \frac{2A}{\pi}$$

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(解法2) で解く

$$a_n = \frac{4}{T} \int_0^{T/2} v(t) \cos n\omega t dt$$

$$= \frac{4}{T} \int_0^{T/2} A \sin \omega t \cos n\omega t dt$$

$2 \cos \alpha \sin \beta = \sin(\alpha + \beta) - \sin(\alpha - \beta)$ を用いて

$$a_n = \frac{2A}{T} \int_0^{T/2} (\sin(n+1)\omega t - \sin(n-1)\omega t) dt$$

$$= \frac{2A}{T} \left(\frac{1}{(n+1)\omega} [-\cos(n+1)\omega t]_0^{T/2} - \frac{1}{(n-1)\omega} [-\cos(n-1)\omega t]_0^{T/2} \right)$$

$$= \frac{2A}{\omega T} \left(-\frac{1}{n+1} [\cos(n+1)\omega t]_0^{T/2} + \frac{1}{n-1} [\cos(n-1)\omega t]_0^{T/2} \right)$$

$$= \frac{2A}{\omega T} \left\{ -\frac{1}{n+1} \left(\cos(n+1) \frac{\omega T}{2} - \cos 0 \right) + \frac{1}{n-1} \left(\cos(n-1) \frac{\omega T}{2} - \cos 0 \right) \right\}$$

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$\omega = \frac{2\pi}{T}$ を用いて

$$a_n = \frac{2A}{2\pi} \left\{ -\frac{1}{n+1} (\cos(n+1)\pi - 1) + \frac{1}{n-1} (\cos(n-1)\pi - 1) \right\}$$

$$a_1 = \frac{A}{\pi} \left\{ -\frac{1}{2} (\cos 2\pi - 1) + \frac{1}{1-1} (\cos 0 - 1) \right\} = 0$$

$$a_2 = \frac{A}{\pi} \left\{ -\frac{1}{3} (\cos 3\pi - 1) + \frac{1}{1} (\cos \pi - 1) \right\} = \frac{A}{\pi} \left\{ -\frac{1}{3} (-2) - 2 \right\} = \frac{A}{\pi} \left(-\frac{4}{3} \right)$$

$$a_3 = \frac{A}{\pi} \left\{ -\frac{1}{4} (\cos 4\pi - 1) + \frac{1}{2} (\cos 2\pi - 1) \right\} = 0$$

$$a_4 = \frac{A}{\pi} \left\{ -\frac{1}{5} (\cos 5\pi - 1) + \frac{1}{3} (\cos 3\pi - 1) \right\} = \frac{A}{\pi} \left\{ -\frac{1}{5} (-2) + \frac{1}{3} (-2) \right\}$$

$$= \frac{A}{\pi} \left(\frac{2}{5} - \frac{2}{3} \right) = \frac{A}{\pi} \left(\frac{6-10}{15} \right) = \frac{A}{\pi} \left(-\frac{4}{15} \right)$$

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よって

$$v(t) = \frac{4A}{\pi} \left(\frac{1}{2} - \frac{1}{3} \cos 2\omega t - \frac{1}{15} \cos 4\omega t \dots \right)$$

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