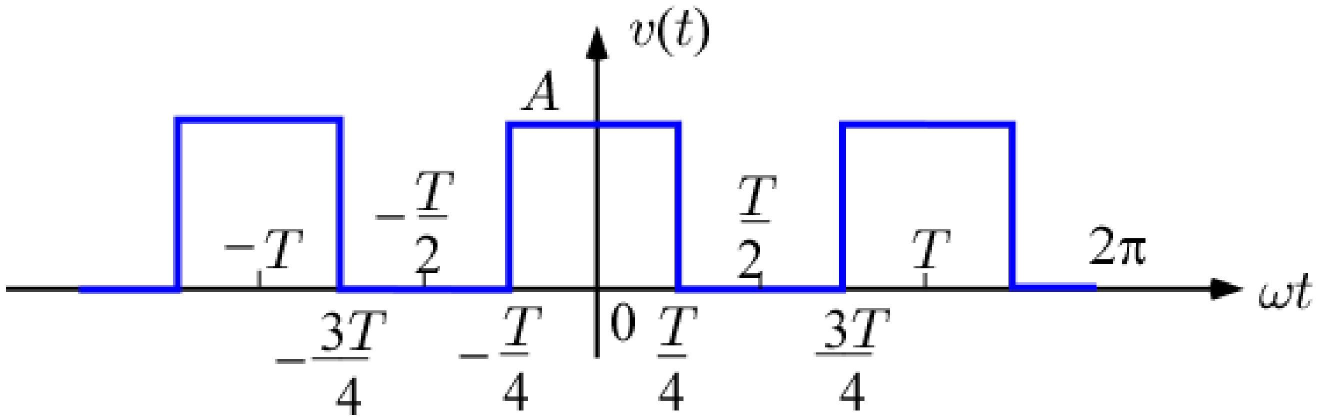
第 24 章：非正弦波交流

キーワード：対称波，奇関数波，偶関数波

学習目標：対称波，奇関数波，偶関数波のフーリエ級数を計算できる。

[問題30.1](1)

$$\omega = \frac{2\pi}{T}$$



$$f(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\omega t + \sum_{n=1}^{\infty} b_n \sin n\omega t$$

$$b_n = \frac{2}{T}\int_0^T v(t) \sin n\omega t \ dt$$

偶関数より $b_n = 0$

a_0 を求める

(解法1)

$$a_0 = \frac{1}{T}\int_0^T v(t)\ dt = \frac{1}{T}\left(\int_0^{\frac{T}{4}} A\ dt + \int_{\frac{3T}{4}}^{T} A\ dt\right)$$

(解法2)半周期を2倍

$$a_0 = \frac{2}{T}\int_0^{\frac{T}{2}} v(t)\ dt = \frac{2}{T}\int_0^{\frac{T}{4}} A$$

（解法3）周期を $-\frac{T}{4} \sim \frac{3T}{4}$ と考える

$$a_0 = \frac{1}{T}\int_{-\frac{T}{4}}^{\frac{3T}{4}} v(t)\ dt\ = \frac{1}{T}\int_{-\frac{T}{4}}^{\frac{T}{4}} A\ dt$$

（解法1）で解く

$$a_0 = \frac{A}{T}\left([t]_0^{\frac{T}{4}} + [t]_{\frac{3T}{4}}^{T}\right) = \frac{A}{T}\left(\frac{T}{4}+\frac{T}{4}\right) = \frac{A}{2}$$

（解法2）で解く

$$a_0 = \frac{2A}{T}[t]_0^{\frac{T}{4}} = \frac{2A}{T}\frac{T}{4} = \frac{A}{2}$$

（解法3）で解く

$$a_0 = \frac{A}{T}[t]_{-\frac{T}{4}}^{\frac{T}{4}} = \frac{A}{T}\left(\frac{T}{4}+\frac{T}{4}\right) = \frac{A}{2}$$

a_n を求める

（解法1）で解く

$$a_n = \frac{2}{T}\int_0^T v(t)\cos n\omega t\ dt$$

$$= \frac{2}{T}\left(\int_0^{\frac{T}{4}} A\cos n\omega t\ dt + \int_{\frac{3T}{4}}^{T} A\cos n\omega t\ dt\right)$$

$$= \frac{2A}{T}\left(\left[\frac{1}{n\omega}\sin n\omega t\right]_0^{\frac{T}{4}} + \left[\frac{1}{n\omega}\sin n\omega t\right]_{\frac{3T}{4}}^{T}\right)$$

$\omega T = 2\pi$ より

$$= \frac{2A}{n\omega T}\left(\sin n\omega\frac{T}{4} \underbrace{- \sin 0}_{=0} + \sin n\omega T - \sin n\omega\frac{3T}{4}\right)$$

$\omega T = 2\pi$ より

$$a_n = \frac{2A}{2\pi n}\left(\sin\frac{2\pi n}{4} + \underbrace{\sin 2\pi n}_{=0} - \sin\frac{6\pi n}{4}\right)$$

$$= \frac{A}{\pi n}\left(\sin\frac{\pi n}{2} \underbrace{- \sin\frac{3\pi n}{2}}_{=\sin\frac{\pi n}{2}}\right) = \frac{2A}{\pi n}\sin\frac{\pi n}{2}$$

$$a_1 = \frac{2A}{\pi}\sin\frac{\pi}{2} = \frac{2A}{\pi}$$

$$a_2 = \frac{2A}{2\pi}\sin\frac{2\pi}{2} = 0$$

$$a_3 = \frac{2A}{3\pi}\sin\frac{3\pi}{2} = -\frac{2A}{3\pi}$$

よって

$$v(t) = \frac{A}{2} + \frac{2A}{\pi}\left(1 \cdot \cos\omega t - \frac{1}{3}\cos 3\omega t + \frac{1}{5}\cos 5\omega t + \cdots\right)$$

$$
\begin{aligned}
b_n &= \frac{2}{T}\int_0^T v(t)\sin n\omega t \; dt \\
&= \frac{2}{T}\left(\int_0^{\frac{T}{4}} A\sin n\omega t \; dt + \int_{\frac{3T}{4}}^{T} A\sin n\omega t \; dt\right) \\
&= \frac{2A}{T}\left(\left[-\frac{1}{n\omega}\cos n\omega t\right]_0^{\frac{T}{4}} + \left[-\frac{1}{n\omega}\cos n\omega t\right]_{\frac{3T}{4}}^{T}\right) \\
&= -\frac{2A}{n\omega T}\left(\cos n\omega\frac{T}{4} - \cos 0 + \cos n\omega T - \cos n\omega\frac{3T}{4}\right) \\
&= -\frac{2A}{2n\pi}\left(\cos n\frac{\pi}{2} - \cos 0 + \cos 2n\pi - \cos n\frac{3\pi}{2}\right)
\end{aligned}
$$

$$
\cos n\frac{\pi}{2} = 0, \quad (n = 1, 2, \cdots) \qquad \cos n\frac{3\pi}{2} = 0, \quad (n = 1, 2, \cdots)
$$

$$
\cos 2n\pi = 1, \quad (n = 1, 2, \cdots)
$$

$$
= 0
$$

（解法2）で解く　＊（解法1）との違いを示す

$$a_n = \frac{2}{T}\int_0^T v(t)\cos n\omega t\ dt$$

$$= \frac{\overset{4}{\cancel{2}}}{T}\cancel{\Bigg(}\int_0^{\frac{T}{4}} A\cos n\omega t\ dt \cancel{+ \int_{\frac{3T}{4}}^{T} A\cos n\omega t\ dt}\cancel{\Bigg)}$$

$$= \frac{\overset{4}{\cancel{2}}A}{T}\cancel{\Bigg(}\left[\frac{1}{n\omega}\sin n\omega t\right]_0^{\frac{T}{4}} \cancel{+ \left[\frac{1}{n\omega}\sin n\omega t\right]_{\frac{3T}{4}}^{T}}\cancel{\Bigg)}$$

$$= \frac{\overset{4}{\cancel{2}}A}{n\omega T}\cancel{\Bigg(}\sin n\omega\frac{T}{4} \underbrace{- \sin 0}_{=0} + \cancel{\sin n\omega T - \sin n\omega\frac{3T}{4}}\cancel{\Bigg)}$$

$\omega T = 2\pi$ より

$$a_n = \frac{\overset{4}{\cancel{2}}A}{2\pi n}\cancel{\Bigg(}\sin\frac{2\pi n}{4} \cancel{+ \sin 2\pi n - \sin\frac{6\pi n}{4}}\cancel{\Bigg)}$$

$$= \frac{\overset{2}{\cancel{A}}}{\pi n}\cancel{\Bigg(}\sin\frac{\pi n}{2} \cancel{- \sin\frac{3\pi n}{2}}\cancel{\Bigg)} = \frac{2A}{\pi n}\sin\frac{\pi n}{2}$$

（解法3）で解く　＊（解法1）との違いを示す

$$a_n = \frac{2}{T}\int_0^T v(t)\cos n\omega t\ dt$$

$$= \frac{2}{T}\left(\int_{\cancel{0}\ -\frac{T}{4}}^{\frac{T}{4}} A\cos n\omega t\ dt + \cancel{\int_{\frac{3T}{4}}^{T} A\cos n\omega t\ dt}\right)$$

$$= \frac{2A}{T}\left(\left[\frac{1}{n\omega}\sin n\omega t\right]_{\cancel{0}\ -\frac{T}{4}}^{\frac{T}{4}} + \cancel{\left[\frac{1}{n\omega}\sin n\omega t\right]_{\frac{3T}{4}}^{T}}\right)$$

$$= \frac{2A}{n\omega T}\left(\sin n\omega\frac{T}{4}\ \cancel{- \sin 0 + \sin n\omega T - \sin n\omega\frac{3T}{4}}\right)$$

$$\frac{-\sin\left(-n\omega\frac{T}{4}\right)}{= +\sin n\omega\frac{T}{4}}$$

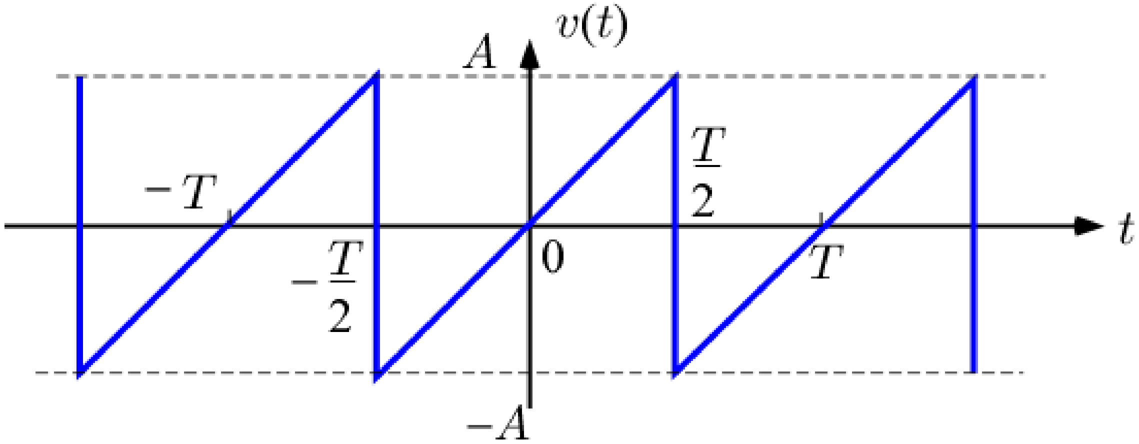
$\omega T = 2\pi$ より

$$a_n = \frac{\cancel{2}^{\,4}A}{2\pi n}\left(\sin\frac{2\pi n}{4}\ \cancel{+ \sin 2\pi n - \sin\frac{6\pi n}{4}}\right)$$

$$= \frac{\cancel{A}^{\,2}}{\pi n}\left(\sin\frac{\pi n}{2}\ \cancel{- \sin\frac{3\pi n}{2}}\right) = \frac{2A}{\pi n}\sin\frac{\pi n}{2}$$

[問題30.1]（2）

$$\omega = \frac{2\pi}{T}$$



$$f(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\omega t + \sum_{n=1}^{\infty} b_n \sin n\omega t$$

奇関数より $a_0 = 0,\ a_n = 0$

b_n を求める

（解法1）

$$b_n = \frac{2}{T}\int_0^T v(t)\sin n\omega t\ dt$$

$$= \frac{2}{T}\left(\int_0^{\frac{T}{2}} \frac{2A}{T}t \sin n\omega t\ dt + \int_{\frac{T}{2}}^{T}\left(\frac{2A}{T}t - 2A\right)\sin n\omega t\ dt\right)$$

(解法2)半周期を2倍

$$b_n = \frac{4}{T}\int_0^{\frac{T}{2}} v(t)\sin n\omega t \; dt \;= \frac{4}{T}\int_0^{\frac{T}{2}} \frac{2A}{T} t \sin n\omega t \; dt$$

(解法3)周期を $-\frac{T}{2} \sim \frac{T}{2}$ と考える

$$b_n = \frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} v(t)\sin n\omega t \; dt \;= \frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} \frac{2A}{T} t \sin n\omega t \; dt$$

b_n を求める

(解法2)で解く

$$b_n = \frac{4}{T}\int_0^{\frac{T}{2}} \frac{2A}{T} t \sin n\omega t \; dt$$

$$= \frac{8A}{T^2}\int_0^{\frac{T}{2}} t \underbrace{\left(-\frac{1}{n\omega}\cos n\omega t\right)'}_{= \sin n\omega t} \; dt$$

部分積分を用いる

$$b_n = \frac{8A}{T^2}\left\{\left[t\left(-\frac{1}{n\omega}\cos n\omega t\right)\right]_0^{\frac{T}{2}} - \int_0^{\frac{T}{2}} -\frac{1}{n\omega}\cos n\omega t\ dt\right\}$$

$$= -\frac{8A}{n\omega T^2}\left\{[t\cos n\omega t]_0^{\frac{T}{2}} - \underline{\int_0^{\frac{T}{2}} \cos n\omega t\ dt}\right\}$$

$$= \left[\frac{1}{n\omega}\sin n\omega t\right]_0^{\frac{T}{2}}$$

$$= -\frac{8A}{n\omega T^2}\left\{\frac{T}{2}\cos n\omega\frac{T}{2} - 0 - \frac{1}{n\omega}\left(\sin n\omega\frac{T}{2} - \sin 0\right)\right\}$$

$\omega T = 2\pi$ より

$$b_n = -\frac{8A}{n2\pi T}\left\{\frac{T}{2}\cos n\pi - \underline{\frac{1}{n\omega}(\sin n\pi)}\right\}$$

$$= 0$$

$$= -\frac{2A}{n\pi}\cos n\pi$$

$$b_1 = -\frac{2A}{\pi}\cos\pi = \frac{2A}{\pi}$$

$$b_2 = -\frac{2A}{2\pi}\cos 2\pi = -\frac{2A}{2\pi}$$

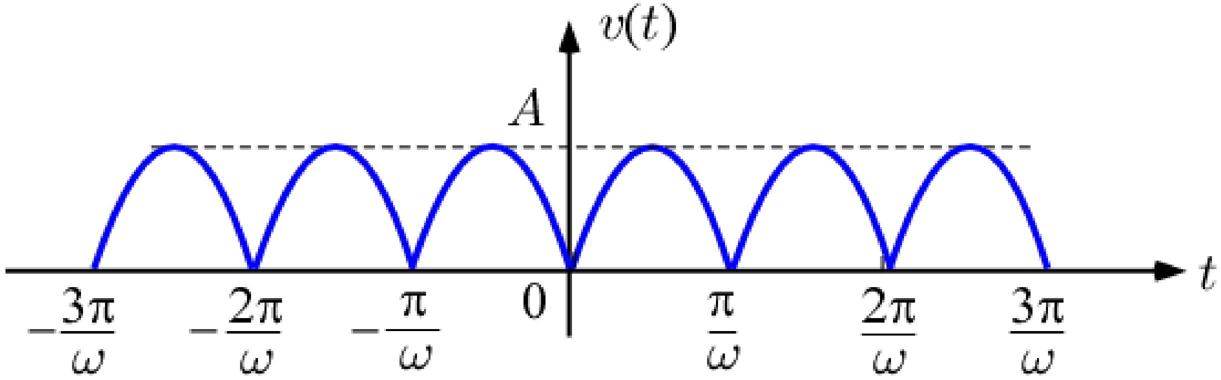
$$b_3 = -\frac{2A}{3\pi}\cos 3\pi = \frac{2A}{3\pi}$$

よって

$$v(t) = \frac{2A}{\pi}\left(\sin\omega t - \frac{1}{2}\sin 2\omega t + \frac{1}{3}\sin 3\omega t + \cdots\right)$$

[問題30.1]（3）

$$\omega = \frac{2\pi}{T}$$



$$f(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\omega t + \sum_{n=1}^{\infty} b_n \sin n\omega t$$

偶関数より $b_n = 0$

a_0 を求める

（解法1）

$$a_0 = \frac{1}{T}\int_0^T v(t)\ dt \quad = \frac{1}{T}\left(\int_0^{\frac{T}{2}} A\sin\omega t\ dt + \int_{\frac{T}{2}}^{T} (-A\sin\omega t)\ dt\right)$$

（解法2）半周期を2倍

$$a_0 = \frac{2}{T}\int_0^{\frac{T}{2}} v(t)\ dt \quad = \frac{2}{T}\int_0^{\frac{T}{2}} A\sin\omega t$$

（解法1）で解く

$$
\begin{aligned}
a_0 &= \frac{1}{T}\left(\int_0^{\frac{T}{2}} A\sin\omega t\ dt + \int_{\frac{T}{2}}^{T}(-A\sin\omega t)\ dt\right) \\
&= \frac{A}{T}\left(\left[-\frac{1}{\omega}\cos\omega t\right]_0^{\frac{T}{2}} + \left[\frac{1}{\omega}\cos\omega t\right]_{\frac{T}{2}}^{T}\right) \\
&= \frac{A}{\omega T}\left([-\cos\omega t]_0^{\frac{T}{2}} + [\cos\omega t]_{\frac{T}{2}}^{T}\right) \\
&= \frac{A}{\omega T}\left(-\cos\frac{\omega T}{2} + \cos 0 + \cos\omega T - \cos\frac{\omega T}{2}\right) \\
&= \frac{A}{2\pi}(-\cos\pi + 1 + \cos 2\pi - \cos\pi) \\
&= \frac{A}{2\pi}(1+1+1+1) \\
&= \frac{2A}{\pi}
\end{aligned}
$$

（解法2）で解く

$$
\begin{aligned}
a_0 &= \frac{2}{T}\int_0^{\frac{T}{2}} A\sin\omega t \\
&= \frac{2A}{T}\left(\left[-\frac{1}{\omega}\cos\omega t\right]_0^{\frac{T}{2}}\right) \\
&= \frac{2A}{\omega T}\left([-\cos\omega t]_0^{\frac{T}{2}}\right) \\
&= \frac{2A}{\omega T}\left(-\cos\frac{\omega T}{2}+\cos 0\right) \\
&= \frac{2A}{2\pi}(-\cos\pi+1) \\
&= \frac{2A}{2\pi}(1+1) \\
&= \frac{2A}{\pi}
\end{aligned}
$$

（解法2）で解く

$$a_n = \frac{4}{T}\int_0^{\frac{T}{2}} v(t)\cos n\omega t\ dt$$

$$= \frac{4}{T}\int_0^{\frac{T}{2}} A\sin\omega t\cos n\omega t\ dt$$

$2\cos\alpha\sin\beta = \sin(\alpha+\beta) - \sin(\alpha-\beta)$ を用いて

$$a_n = \frac{2A}{T}\int_0^{\frac{T}{2}} \left(\sin(n+1)\omega t - \sin(n-1)\omega t\right)\ dt$$

$$= \frac{2A}{T}\left(\frac{1}{(n+1)\omega}\left[-\cos(n+1)\omega t\right]_0^{\frac{T}{2}} - \frac{1}{(n-1)\omega}\left[-\cos(n-1)\omega t\right]_0^{\frac{T}{2}}\right)$$

$$= \frac{2A}{\omega T}\left(-\frac{1}{n+1}\left[\cos(n+1)\omega t\right]_0^{\frac{T}{2}} + \frac{1}{n-1}\left[\cos(n-1)\omega t\right]_0^{\frac{T}{2}}\right)$$

$$= \frac{2A}{\omega T}\left\{-\frac{1}{n+1}\left(\cos(n+1)\frac{\omega T}{2} - \cos 0\right) + \frac{1}{n-1}\left(\cos(n-1)\frac{\omega T}{2} - \cos 0\right)\right\}$$

$\omega = \dfrac{2\pi}{T}$ を用いて

$$a_n = \frac{2A}{2\pi}\left\{-\frac{1}{n+1}\left(\cos(n+1)\pi - 1\right) + \frac{1}{n-1}\left(\cos(n-1)\pi - 1\right)\right\}$$

$$a_1 = \frac{A}{\pi}\left\{-\frac{1}{2}\left(\cos 2\pi - 1\right) + \frac{1}{1-1}\left(\cos 0 - 1\right)\right\} = 0$$

$$a_2 = \frac{A}{\pi}\left\{-\frac{1}{3}\left(\cos 3\pi - 1\right) + \frac{1}{1}\left(\cos \pi - 1\right)\right\} = \frac{A}{\pi}\left\{-\frac{1}{3}(-2) - 2\right\} = \frac{A}{\pi}\left(-\frac{4}{3}\right)$$

$$a_3 = \frac{A}{\pi}\left\{-\frac{1}{4}\left(\cos 4\pi - 1\right) + \frac{1}{2}\left(\cos 2\pi - 1\right)\right\} = 0$$

$$a_4 = \frac{A}{\pi}\left\{-\frac{1}{5}\left(\cos 5\pi - 1\right) + \frac{1}{3}\left(\cos 3\pi - 1\right)\right\} = \frac{A}{\pi}\left\{-\frac{1}{5}(-2) + \frac{1}{3}(-2)\right\}$$

$$= \frac{A}{\pi}\left(\frac{2}{5} - \frac{2}{3}\right) = \frac{A}{\pi}\left(\frac{6-10}{15}\right) = \frac{A}{\pi}\left(\frac{-4}{15}\right)$$

よって

$$v(t) = \frac{4A}{\pi}\left(\frac{1}{2} - \frac{1}{3}\cos 2\omega t - \frac{1}{15}\cos 4\omega t \cdots\right)$$

第 24 章：非正弦波交流

24.1　非正弦波交流

24.2　正弦波の組み合わせと波形

24.3　フーリエ級数による非正弦波の展開

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学習目標：対称波，奇関数波，偶関数波のフーリエ級数を計算できる。